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CALIBER, PRECALIBER, AND THE SHANIN NUMBER OF HYPERSPACES

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Abstract. In the paper, we investigate relationships between cardinal invariants of a topological space X and its hyperspace $\exp X$. We prove that the hyperspace construction preserves caliber, precaliber, and the Shanin numbers.

Keywords: cardinal, topological space, caliber, precaliber, the Shanin number.

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1. Introduction

For the topological space X, we denote

$$\exp X = \{F : F \subset X,\ F \neq \emptyset,\ F \text{ is a closed subset of } X\}.$$

Consider the family $\mathcal{B}$ of all sets in the form of

$$O\langle U_1, \ldots, U_n\rangle = \left\{F \in \exp X : F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset,\ i = 1, 2, \ldots, n\right\},$$

where $U_1, U_2, \ldots, U_n$ are arbitrary open sets in X. The family $\mathcal{B}$ generates the topology on the set $\exp X$. This topology is called *the Vietoris topology.* The set $\exp X$ with the Vietoris topology is called *the exponential space* or *hyperspace* of the space X.

Let X be a topological space. We denote by $\exp_n X$ the family of all non-empty closed subsets of the space X of the cardinality not greater than cardinal number n, i.e. $\exp_n X = \{F \in \exp X : |F| \leq n\}$. Put $\exp_c X = \{F \in \exp X : F \text{ is a compact in } X\}$, $\exp_\omega X = \cup\{\exp_n X : n = 1, 2, \ldots\}$. It is clear that

$$\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X.$$

It is not difficult to see that $\exp_\omega X$ is everywhere dense in $\exp X$, hence $\exp_c X$ is also everywhere dense in $\exp X$.

A cardinal number τ is regular if it cannot be presented in the form of the sum of less than τ cardinal numbers taken in the amount less than τ [3].

A regular cardinal $\tau > \aleph_0$ is said to be *caliber* of the space X if for any family $\mu = \{U_\alpha : \alpha \in A\}$ of non-empty open in X sets such that $|A| = \tau$, there exists $B \subset A$ such that $|B| = \tau$ and $\cap\{U_\alpha : \alpha \in B\} \neq \emptyset$ [3].

Put $k(X) = \{\tau : \tau \text{ is a caliber for } X\}$.

A regular cardinal $\tau > \aleph_0$ is said to be *a precaliber* of the space X if for any family $\mu = \{U_\alpha : \alpha \in A\}$ of non-empty open in X sets such that $|A| = \tau$, there exists $B \subset A$ such that $|B| = \tau$ and $\{U_\alpha : \alpha \in B\}$ is centered [1].

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Put $pk(X) = \{\tau : \tau$ is a precaliber for $X\}$.

The following classical results belong to N.A. Shanin [6].

Theorem 1.1. *If a regular cardinal τ is a precaliber (caliber) of the space X_α for any $\alpha \in A$, then τ is also a precaliber (caliber) of the space $X = \prod\{X_\alpha : \alpha \in A\}$.*

Theorem 1.2. *If $f : X \to Y$ be a mapping "onto" and τ is a precaliber (caliber) of the space X, then τ is also a precaliber (caliber) of the space Y.*

In 1922 L. Vietoris [7] proved the following

Theorem 1.3. *Let X be an infinite compact. Then*

$$w(X) = w(\exp X).$$

In 1957 E. Michael [5] proved the following

Theorem 1.4. *Let X be an infinite T_1-space. Then*
1) $d(X) = d(\exp X)$,
2) $w(X) = w(\exp_c X)$.

In 1997 S. Todorchevic and V. Fedorchuk [4] proved the following

Theorem 1.5. *Let X be an infinite compact. Then*

$$c(\exp X) = \sup\{c(X^n) : n \in \mathbb{N}\}.$$

It is known that any everywhere dense subset inherits the Souslin number, and therefore theorem 1.5 is true for any infinite Tychonoff space X since it is everywhere dense in its Stone-Čech compact extension, i.e. the following result [4] takes place:

Theorem 1.6. *Let X be an infinite Tychonoff space. Then*

$$c(\exp X) = \sup\{c(X^n) : n \in \mathbb{N}\}.$$

2. Main results

Proposition 2.1. *Let Y be everywhere dense in X and $\tau \in k(Y)$ is a caliber of the subspace Y. Then τ is a caliber of the space X.*

Proof. : Let $\tau \in k(Y)$, and $\mu = \{U_\alpha : \alpha \in A\}$ be a family of non-empty open in X sets such that $|A| = \tau$. Put $\mu' = \{U_\alpha \cap Y = U'_\alpha : \alpha \in A\}$. It is a family of non-empty open subsets in Y. Since $\tau \in k(Y)$, there exists a subset $B \subset A$ and $|B| = \tau$ such that $\emptyset \neq \{U'_\alpha : \alpha \in B\} \subset \{U_\alpha : \alpha \in B.\}$. So, $\cap\{U_\alpha : \alpha \in B\} \neq \emptyset$ and τ is a caliber of the space X. □

Corollary 2.1. *Let $\tau \in k(\exp_\omega X)$. Then τ is a caliber of the spaces $\exp_c X$ and $\exp X$.*

Proposition 2.2. *Let X be an infinite T_1-space and $\tau \in k(\exp X)$. Then $\tau \in k(X)$.*

Proof. : Let $\tau \in k(\exp X)$ and $\mu = \{U_\alpha : \alpha \in A\}$ be a family of arbitrary open subsets of the space X of the cardinality $|A| = \tau$. Put

$$\langle\mu\rangle = \{O\langle U_\alpha\rangle : \alpha \in A\}.$$

Since τ is a caliber of the space $\exp X$, then there exists a subfamily $B \subset A$ such that $\cap\{O\langle U_\alpha\rangle : \alpha \in B\} \neq \emptyset$. Let $F \in \{O\langle U_\alpha\rangle : \alpha \in B\}$. Then $F \in \langle U_\alpha\rangle$ for all $\alpha \in B$. It means that $F \subset U_\alpha$ for each $\alpha \in B$. Choose an arbitrary point $x \in F$. Then $x \in \cap\{U_\alpha : \alpha \in B\}$ and $|B| = \tau$. Hence τ is a caliber of the space X. □

Theorem 2.1. *Let X be an infinite T_1-space. Then*

$$k(X) = k(\exp_n X) = k(\exp_\omega X) = k(\exp_c X) = k(\exp X).$$

Proof. We'll show that $k(X) \subset k(\exp_n X) \subset k(\exp_\omega X) \subset k(\exp_c X) \subset k(\exp X)$. Let τ is a caliber for the space X. Then by theorem 1.1 τ is a caliber of the space X^n. On the other hand, if $f : X \to Y$ is a continuous mapping "onto" and τ is a caliber of the space X, then τ is a caliber of the space $f(X)$. Hence $k(X) \subset k(\exp_n X)$. The inclusion $k(\exp_n X) \subset k(\exp_\omega X)$ is evident. By virtue of proposition 2.1 and corollary 2.1 we have $k(\exp_\omega X) \subset k(\exp_c X) \subset k(\exp X)$. The inclusion $k(\exp X) \subset k(X)$ follows from proposition 2.2. □

Proposition 2.3. *Let Y be everywhere dense in X. Then $pk(Y) \subset pk(X)$.*

Proof. Let $\tau \in pk(Y)$ and $\mu = \{U_\alpha : \alpha \in A, |A| = \tau\}$ be a family of non-empty open subsets of the space X. Put $\mu' = \{U_\alpha \cap Y = U'_\alpha : \alpha \in A, |A| = \tau\}$. Since $\tau \in pk(Y)$, there exists a subfamily $B \subset A$, $|B| = \tau$, such that the family $\{U'_\alpha : \alpha \in B\}$ is centered. But $\{U'_\alpha : \alpha \in B\} \subset \{U_\alpha : \alpha \in B\}$. Hence, the family $\{U_\alpha : \alpha \in B\}$ is centered in the space X. □

Corollary 2.2. *Let $\tau \in pk(\exp_\omega X)$. Then τ is a precaliber of the spaces $\exp_c X$ and $\exp X$.*

Proposition 2.4. *Let $\tau \in pk(\exp X)$. Then $\tau \in pk(X)$.*

Proof. Let $\tau \in pk(\exp X)$ and $\mu = \{U_\alpha : \alpha \in A, |A| = \tau\}$ be a family of non-empty open subsets of the space X of the cardinality τ. Then $\langle\mu\rangle = \{O\langle U_\alpha\rangle : \alpha \in A, |A| = \tau\}$ is a family of open subsets of the space $\exp X$ of the cardinality τ. By the condition, $\tau \in pk(\exp X)$, therefore there exists a subfamily $B \subset A$ such that the family $\langle\mu_1\rangle = \{O\langle U_\alpha\rangle : \alpha \in B, |B| = \tau\}$ is centered in $\exp X$. Let $U_1, \ldots, U_n \in \mu_1$ be arbitrary finite sets from the family μ_1. Then $O\langle U_1\rangle, \ldots, O\langle U_n\rangle$ is a centered system in $\exp X$, i.e. $\cap\{O\langle U_i\rangle : i = 1, \ldots, n\} \neq \emptyset$. Then $F \subset \cap\{U_i : i = 1, \ldots, n\}$. Hence, the family μ_1 is centered, i.e. $\tau \in pk(X)$. □

Theorem 2.2. *Let X be an infinite T_1-space. Then*

$$pk(X) = pk(\exp_n X) = pk(\exp_\omega X) = pk(\exp_c X) = pk(\exp X).$$

Proof. Let us prove the inclusion

$$pk(X) \subset pk(\exp_n X) \subset pk(\exp_\omega X) \subset pk(\exp_c X) \subset pk(\exp X).$$

1) Let $\tau \in pk(X)$. Then by theorem 1.1 we have $\tau \in pk(X^n)$. The space $\exp_n X$ is the continuous image of the space X^n, therefore by theorem 1.2 $\tau \in pk(\exp_n X)$ for each $n \in \mathbb{N}$.

2) Let us prove the inclusion $pk(\exp_n X) \subset pk(\exp_\omega X)$. Let $\tau \in pk(\exp_n X)$ for all $n \in \mathbb{N}$. We'll show that $\tau \in pk(\exp_\omega X)$. Let $\mu = \{O_\alpha = O_\alpha\langle U_1, \ldots, U_n\rangle : \alpha \in A\}$ be a family of non-empty open subsets in $\exp_\omega X$ of the cardinality $|A| = \tau$. There exists a space $\exp_k X$ such that $\mu \cap \exp_k X = \{O'_\alpha = O_\alpha \cap \exp_k X \neq \emptyset : \alpha \in B \subset A\}$ and $|B| = \tau$. Since the cardinal $\tau \in pk(exp_k X)$, then there exists $B' \subseteq B$ such that $|B'| = \tau$ and the system $\{O'_\alpha : \alpha \in B'\}$ is centered. It is clear that $\{O'_\alpha \subset O_\alpha : \alpha \in B' \subset B \subset A\}$. Hence, the system $\{O_\alpha : \alpha \in B' \subset A\}$ is centered, i.e. $\tau \in pk(\exp_\omega X)$.

3) The inclusions $pk(\exp_\omega X) \subset pk(\exp_c X)$ and $pk(\exp_c X) \subset pk(\exp_\omega X)$ follow from proposition 2.3 and corollary 2.2.

4) The inclusion $pk(\exp X) \subset pk(X)$ follows from proposition 2.4. □

Now we introduce the Shanin number $sh(X)$.

The cardinal $\min\{\tau : \tau^+$ is a caliber of $X\}$ is called the Shanin number of the space X and denoted by $sh(X)$.

Further, $psh(X) = \min\{\tau : \tau^+$ is a precaliber of $X\}$. The following inequalities always take place:

$$c(X) \leq psh(X) \leq sh(X) \leq d(X).$$

Theorem 2.3. *Let X be an infinite T_1-space. Then*

$$sh(X) = sh(\exp_n X) = sh(\exp_\omega X) = sh(\exp_c X) = sh(\exp X).$$

Proof. Let X be an infinite T_1-space and $sh(X) = \tau \geq \aleph_0$. By definition of the Shanin number, τ^+ is a regular cardinal and it is also a caliber for the space X. By virtue of theorem 2.1, $k(X) = k(\exp_n X) = k(\exp_\omega X) = k(\exp_c X) = k(\exp X) = \tau^+$. Hence, by definition of the Shanin number, we have that $sh(X) = sh(\exp_n X) = sh(\exp_\omega X) = sh(\exp_c X) = sh(\exp X)$. □

Corollary 2.3. *Let X be an infinite T_1-space. Then the spaces X, $\exp_n X$, $\exp_\omega X$, $\exp_c X$ $\exp X$ satisfy the Shanin condition simultaneously.*

Theorem 2.2 implies immediately the following

Theorem 2.4. *Let X be an infinite T_1-space. Then*

$$psh(X) = psh(\exp_n X) = psh(\exp_\omega X) = psh(\exp_c X) = psh(\exp X).$$

Proposition 2.5. *Let $d(X) = \tau \geq \aleph_0$. Then τ^+ is a caliber of the space X.*

Proof. Suppose the opposite, let there exists a family $\mu = \{U_\alpha : \alpha \in A, |A| = \tau\}$ of open in X subsets such that one cannot choose from it a subfamily $B \subset A$ of the cardinality τ^+ such that $\cap\{U_\alpha : \alpha \in B\} \neq \emptyset$. Then one can choose a disjunct family of open subsets of the cardinality τ^+, what contradicts the inequality $c(X) \leq d(X)$ for each topological space X. Hence, τ^+ is a caliber of the space X. □

Corollary 2.4. *Let X be a separable space. Then each infinite cardinal is its caliber.*

Recall that a topological space X has a weak density $\leq \tau$ [2] if τ is the smallest infinite cardinal such that there exists a π-base of X which is an union of τ centered systems. In this case we write $wd(X) \leq \tau$. A space X is said to be weakly separable if $wd(X) \leq \aleph_0$.

Proposition 2.6. *Let $wd(X) = \tau \leq \aleph_0$. Then τ^+ is a precaliber of the space X.*

Proof. Let $wd(X) = \tau \leq \aleph_0$. Then by theorem 1.4 [2] there exists a τ-dense extension eX, i.e. $d(eX) = \tau$. By virtue of Proposition 2.5, we have τ^+ is a caliber of the space eX. The space X is everywhere dense in eX, therefore by proposition 2.1, τ^+ is a precaliber for X. □

Corollary 2.5. *Let X be a weakly separable space. Then every uncountable cardinal is a precaliber of the space X.*

Proposition 2.7. *Let X be any topological T_1-space. Then*

$$psh(X) \leq wd(X).$$

Proof. Let $wd(X) = \tau$. Then there exists in X a π-base decomposed on τ centered families of open sets, i.e. $B = \cup\{B_\alpha : \alpha \in A, |A| = \tau\}$ is a π-base and every $B_\alpha = \{U_s^\alpha : s \in A_\alpha\}$ is a centered system of open sets for each $\alpha \in A$. Let $\mu = \{G_\beta : \beta \in M, |M| = \tau^+\}$ be an arbitrary family of open sets in the space X. The system B is a π-base in X. Then for each G_β, one can find $U_s^\alpha \in B_\alpha$ such that $U_s^\alpha \subset G_\beta$. Further, one can find a number $\alpha \in A$ such that the family of open sets $\{U_s^\alpha : \alpha \in A_\alpha\}$ of the cardinality τ^+ is in the system $\{G_\beta : \beta \in M', |M'| = \tau^+\}$. Since the system $\{U_s^\alpha : \alpha \in A_\alpha\}$ is centered, then the system $\{G_\beta : \beta \in M', |M'| = \tau^+\}$ is also centered. Hence, $psh(X) \leq \tau$. □

Theorem 2.5. *Let $wd(X_\alpha) = \tau \geq \aleph_0$ for each $\alpha \in A$. Then τ^+ is a precaliber of the product $\prod\{X_\alpha : \alpha \in A\}$.*

Proof. Let $wd(X_\alpha) = \tau \geq \aleph_0$ for each $\alpha \in A$. Then by proposition 2.6 we have that τ^+ is a precaliber of the space X_α for each $\alpha \in A$. By virtue of theorem 1.1, we have that τ^+ is a precaliber of the product $\prod\{X_\alpha : \alpha \in A\}$. □

Corollary 2.6. *If X_α be a weakly separable space for each $\alpha \in A$, then any uncountable cardinal is a precaliber of the product $\prod\{X_\alpha : \alpha \in A\}$.*

Corollary 2.7. *If X_α be a weakly separable space for each $\alpha \in A$ then the product $\prod\{X_\alpha : \alpha \in A\}$ satisfies the Souslin condition, i.e. $c\left(\prod\{X_\alpha : \alpha \in A\}\right) \leq \aleph_0$.*

Proposition 2.8. *Let τ be an infinite cardinal and τ^+ is a precaliber (a caliber) for the space X. Then the Souslin number $c(X) \leq \tau$.*

Proof follows immediately from the definition of the Souslin number.

Corollary 2.8. *Let $\aleph_1$ be a precaliber for the space X. Then X satisfies the Souslin condition.*

Theorem 2.5 and proposition 2.7 imply the following

Theorem 2.6. *Let $d(X_\alpha) = \tau \geq \aleph_0$ for each $\alpha \in A$. Then τ^+ is a precaliber of the product $\prod\{X_\alpha : \alpha \in A\}$.*

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